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STEPS TOWARD A CONSTRUCTIVE NOMINALISM

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1. Renunciation of abstract entities. We do not believe in abstract entities. No one supposes that abstract entities—classes, relations, properties, etc.—exist in space-time; but we mean more than this. We renounce them altogether.

We shall not forego all use of predicates and other words that are often taken to name abstract objects. We may still write 'x is a dog,' or 'x is between y and z'; for here 'is a dog' and 'is between . . . and' can be construed as syncategorematic: significant in context but naming nothing. But we cannot use variables that call for abstract objects as values.[1] In 'x is a dog,' only concrete objects are appropriate values of the variable. In contrast, the variable in 'x is a zoölogical species' calls for abstract objects as values (unless, of course, we can somehow identify the various zoölogical species with certain concrete objects). Any system that countenances abstract entities we deem unsatisfactory as a final philosophy.

Renunciation of abstract objects may leave us with a world composed of physical objects or events, or of units of sense experience, depending upon decisions that need not be made here. Moreover, even when a brand of empiricism is maintained which acknowledges repeatable sensory qualities as well as sensory events,[2] the philosophy of mathematics still faces essentially the same problem that it does when all universals are repudiated. Mere sensory qualities afford no adequate basis for the unlimited universe of numbers, functions, and other classes claimed as values of the variables of classical mathematics.

Why do we refuse to admit the abstract objects that mathematics needs? Fundamentally this refusal is based on a philosophical intuition that cannot be justified by appeal to anything more ultimate. It is fortified, however, by certain *a posteriori* considerations. What seems to be the most natural principle for abstracting classes or properties leads to paradoxes. Escape from these paradoxes can apparently be effected only by recourse to alternative rules whose artificiality and arbitrariness arouse suspicion that we are lost in a world of make-believe.[3]

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[1] That it is in the values of the variables, and not in the supposed designata of constant terms, that the ontology of a theory is to be sought, has been urged by W. V. Quine in *Notes on existence and necessity*, ***The journal of philosophy***, vol. 40 (1943), pp. 113–127; also in *Designation and existence*, ibid., vol. 36 (1939), pp. 701–709.

[2] As for example in Nelson Goodman's ***A study of qualities*** (1941, typescript, Harvard University Library). Qualitative ("abstract") particles of experience and spatio-temporally bounded ("concrete") particles are there regarded as equally acceptable basic elements for a system. Devices described in the present paper will probably make it possible so to revise that study that no construction will depend upon the existence of classes.

[3] The simple principle of class abstraction, which leads to Russell's paradox and others, is this: Given any formula containing the variable 'x', there is a class whose members are all and only the objects x for which that formula holds. See W. V. Quine, ***Mathematical logic***, pp. 128–130. For a brief survey of systems designed to exclude the paradoxes, see pp. 163–166, op. cit.; also *Element and number*, this JOURNAL, vol. 6 (1941), pp. 135–149.

2. Renunciation of infinity. We decline to assume that there are infinitely many objects. Not only is our own experience finite, but there is no general agreement among physicists that there are more than finitely many physical objects in all space-time.[4] If in fact the concrete world is finite, acceptance of any theory that presupposes infinity would require us to assume that in addition to the concrete objects, finite in number, there are also abstract entities.

Classical arithmetic presupposes an infinite realm of numbers. Hence if, in an effort to reconcile arithmetic with our renunciation of abstract entities, we were to undertake to identify numbers arbitrarily with certain things in the concrete world, we should thereby drastically curtail classical arithmetic; for, we cannot assume there are infinitely many such things.

Classical syntax, like classical arithmetic, presupposes an infinite realm of objects; for it assumes that the expressions it treats of admit concatenation to form longer expressions without end. But if expressions must, like everything else, be found within the concrete world, then a limitless realm of expressions cannot be assumed. Indeed, expressions construed in the customary way as abstract typographical shapes do not exist at all in the concrete world; the language elements in the concrete world are rather inscriptions or marks, the shaped objects rather than the shapes.[5]

The stock of available inscriptions can be vastly increased if we include, not only those which have colors or sounds contrasting with the surroundings, but all appropriately shaped spatio-temporal regions even though they be indistinguishable from their surroundings in color, sound, texture, etc. But the number and length of inscriptions will still be limited insofar as the spatio-temporal world itself is limited. Consequently we cannot say that in general, given any two inscriptions, there is an inscription long enough to be the concatenation of the two.

Furthermore, there can be at most only as many inscriptions as concrete objects. Hence, if concrete objects are finite in number, there are bound to be some for which there are no names or descriptions whatever. Otherwise every concrete object would have to be the name or description of a unique and distinct concrete object; and we should thus be deprived of all predicates and connectives, to say nothing of synonyms, duplicate inscriptions, and non-inscriptions.

3. The nominalist's problems. By renouncing abstract entities, we of course exclude all predicates which are not predicates of concrete individuals or explained in terms of predicates of concrete individuals. Moreover, we reject any statement or definition—even one that explains some predicates of concrete individuals in terms of others—if it commits us to abstract entities. For example, until we find some way of construing 'is an ancestor of' in terms of 'is a

[4] According to quantum physics, each physical object consists of a finite number of spatio-temporally scattered quanta of action. For there to be infinitely many physical objects, then, the world would have to have infinite extent along at least one of its spatio-temporal dimensions. Whether it has is a question upon which the current speculation of physicists seems to be divided.

[5] A nominalistic syntax language may, of course, still contain shape-predicates, enabling us to say that a given inscription is, for example, dot-shaped, dotted-line-shaped, Odyssey-shaped. See §5 and §10.

parent of' other than the way the ancestral of a relation is usually defined in systems of logic,[6] the relationship between these predicates remains for us unexplained.

We shall, then, face problems of reducing predicates of abstract entities to predicates of concrete individuals, and also problems of constructing certain predicates of concrete individuals in terms either of certain others or of any others that satisfy some more or less well-defined criteria. Apart from those predicates of concrete objects which are permitted by the terms of the given problem to appear in the definiens, nothing may be used but individual-variables, quantification with respect to such variables, and truth-functions. Devices like recursive definition and the notion of ancestral must be excluded until they themselves have been satisfactorily explained.

We are not *as nominalists* concerned with the motives behind the demand that a given predicate of concrete individuals be defined in terms of certain other such predicates. Naturally the demand may often arise from a feeling that the latter predicates are in some sense the clearer, and we may as persons often share this feeling; but purely as nominalists we know no differences of clarity among predicates of concrete individuals.[7] Our problem is solely to provide, where definitions are called for, definitions that are free of any terms or devices that are tainted by belief in the abstract. We shall naturally first try to find definitions where, for varied reasons, we feel they are most urgently needed; and we shall not waste time looking for definitions in terms of predicates that we suppose to be ambiguous or self-contradictory. But, as has perhaps been illustrated by the case of 'ancestor' and 'parent,' it cannot be said that the explanation of one predicate in terms of another is of interest only if the latter is regarded as clearer. Indeed, if we have only a pseudo-explanation (involving abstract entities) relating predicates of individuals, the problem of replacing it by a genuine construction has as immediate interest as the problem of defining a given predicate in terms of others which come up to a certain standard of clarity, or the problem of explaining a predicate of abstract entities.

4. Some nominalistic reductions. Some statements that seem to be about abstract entities can be rephrased in well-known ways as statements about concrete objects. Thus, where 'A' and 'B' are thought of as fixed terms and not as bindable variables, the statement:

Class A is included in Class B

may be rephrased as:

Everything that is an A is a B.

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[6] The usual definition, which was first set forth by Frege in 1879 (***Begriffschrift***, p. 60), has become well-known through Whitehead and Russell and other writers. It is presented once more in the next section.

[7] It might be supposed that the nominalist must regard as unclear any predicate of individuals for which there is no explanation that does not involve commitment to abstract entities. But unless "explanation" as here intended depends upon standards of clarity, which do not concern the nominalist as nominalist, a suitable explanation can always be supplied trivially by equating the predicate in question with any arbitrarily concocted single word.

The phrases 'is an A' and 'is a B' here are predicates of concrete objects, and are regarded as naming nothing in themselves; that is to say, the positions which they occupy are treated as inaccessible to bound variables.

Certain statements which even involve explicit quantification over classes are replaceable by equivalent statements which conform to the tenets of nominalism. To take a simple example, the statement:

Class A is included in some class other than A

is equivalent to:

Something is not an A.

Statements purporting to specify sizes of finite classes of concrete objects are also easily accommodated. Thus the statement:

Class A has three members

may be rendered:

There are distinct objects x, y, and z such that anything is an A if and only if it is x or y or z;

i.e.:

$$(\exists x)(\exists y)(\exists z)(x \neq y \,.\, y \neq z \,.\, x \neq z \,.\, (w)(Aw \equiv: w = x \,.\!\vee\!.\, w = y \,.\!\vee\!.\, w = z))\cdot$$

Obviously any statement affirming or denying that there are just, or at least, or at most, a certain number of concrete individuals satisfying a given predicate can be readily translated in similar fashion, provided the translation is short enough to fit into the universe.[8]

The definition of ancestorhood in terms of parenthood according to Frege's method seems to involve a class-variable even more essentially. The definiens of 'b is an ancestor of c' would run thus:

b is distinct from c; and, for every class x, if c is a member of x and all parents of members of x are members of x then b is a member of x;

i.e.:

$$b \neq c \,.\, (x)\{c \in x \,.\, (y)(z)(z \in x \,.\, \text{Parent } yz \,.\!\supset\!.\, y \in x) \,.\!\supset\!.\, b \in x\}.$$

But we can translate this sentence also with help of the notation 'Part st,' meaning that the individual s is part (or all) of the individual t.[9] We need only re-

[8] The nominalist need not necessarily regard such a sentence as 'There are 10^{1000} objects in the universe' as meaningless, even though there be no translation along these lines. For, this sentence can be translated as 'The universe (as an individual) has 10^{1000} objects as parts' where 'has 10^{1000} objects as parts' is taken as a primitive predicate of individuals. But while this translation satisfies purely nominalistic demands, there may be extra-nominalistic reasons of economy or clarity for wanting a translation that contains no such predicate. And wherever and for whatever reasons a translation of an expression is wanted in terms of certain predicates or a certain kind of predicates, the search for such a translation is a problem for the nominalist—though of course neither he nor any one else claims that every predicate can be defined in terms of every possible set of others.

[9] A systematic treatment of 'part' and kindred terms will be found in *The calculus of individuals and its uses* by Henry S. Leonard and Nelson Goodman in this JOURNAL, vol. 5

place 'class' by 'individual,' and 'member' by 'part,' provided we also stipulate that b be a parent and c have a parent. This added stipulation insures that b and c be single whole organisms, rather than fragments or sums of organisms. In symbols, 'b is an ancestor of c' becomes:

$$b \neq c \,.\, (\exists u) \text{ Parent } bu \,.\, (\exists w) \text{ Parent } wc \,.\, (x)\{\text{Part } cx \,. \\ (y)(z)(\text{Part} zx . \text{Parent } yz \,.\supset \text{ Part } yx) \,.\supset \text{ Part } bx\}.$$

Clearly the above method of translation presupposes that an individual may be spatio-temporally scattered, or discontinuous. It presupposes that continuity is not necessary for concreteness. A broken dish is no less concrete than a whole one, but merely has more complicated boundaries; and any totality of individuals, however disperse in space and time, counts as an individual in turn. Individuals, thus liberally construed, serve some of the purposes of classes, as is evident from the above treatment of 'ancestor.' But it is by no means true that we can in general simply identify any class of individuals with a scattered single individual, and reconstrue 'member' as 'part.' The individual composed of all persons, e.g., has many parts which are not persons; some of these parts are parts of persons, and some consist of many persons or of parts of many persons. In the above analysis of 'ancestor,' we were able to overcome this difficulty by inserting the clause '$(\exists u)$ Parent bu . $(\exists w)$ Parent wc.' Commonly, however, this kind of difficulty admits of no such simple solution.

The two-place predicate 'is ancestor of' is, to borrow terminology from the platonistic logic of relations, the (proper) *ancestral* of the two-place predicate 'is parent of.' We have seen, above, how it can be defined. But the scheme used there does not work for the ancestral of every two-place predicate of individuals. It works so long as every individual has at least some part which it shares with none of the individuals which are its "ancestors" (with respect to the predicate in question). At the present writing we know no way of defining the ancestral of *every* two-place predicate of individuals nominalistically.

A rather different problem is raised by such statements as:

There are more cats than dogs.

As pointed out earlier, we are already able to deal with such statements as 'There is at least one cat and not at least one dog' and 'There are at least two cats and not at least two dogs.' An alternation of enough such successive statements will be true if and only if there are more cats than dogs, because it will contain at least one component statement that is true in view of the actual number of cats and of dogs. Use of this method requires, first, knowledge that in all space-time there are not more than so many (say fifty trillion) dogs, and second, a prodigious amount of writing or talking. Even though the requisite knowledge be available, the practical difficulties of actually writing or speaking the translation of the statement about cats and dogs would be prohibitive.

(1940), pp. 45–55. Earlier versions were published by Tarski and Leśniewski. Although all of these would have to undergo revision to meet the demands of nominalism, such revision is for the most part easily accomplished and does not affect any of the uses to which the terms in question are put here.

A better method of translation makes use of the predicate 'is part of' and another simple auxiliary predicate: 'is bigger than.' The predicate 'is a bit' is then so defined that it applies to every object that is just as big as the smallest animal among all cats and dogs. In other words, 'x is a bit' is defined to mean that for every y, if y is a cat or a dog and is bigger than no other cat or dog, then neither is x bigger than y nor is y bigger then x. For brevity we shall call x a *bit of z* when x is a bit and is part of z. Now if and only if there are more cats than dogs will it be the case that every individual that contains at least one bit of each cat is bigger than some individual that contains at least one bit of each dog. (Such an individual will of course be spatio-temporally scattered.) Accordingly we may translate the sentence 'There are more cats than dogs' as follows:

Every individual that contains a bit of each cat is bigger than some individual that contains a bit of each dog.

(Symbolic transcriptions are omitted here, as they will be given later for parallel cases: §6, D9-10.)

This method of translation has the great advantage, over the first method suggested, that there is no practical difficulty about writing down an actual translation, regardless of the multiplicity of individuals concerned. But, like our method of defining the ancestral, it is not completely general. It will still work if, in place of 'is a cat' and 'is a dog,' we choose any other two predicates each of which is such that the individuals fulfilling it are discrete from one another. Thus it holds good for such a case as:

There are more human cells than humans,

and indeed for most cases where such numerical comparisons are made in ordinary discourse. It has an important use in nominalistic syntax, as we shall see later. Moreover, by a relatively simple change it can be made general enough to work wherever each individual fulfilling either of the two predicates has a part that it shares with no other individual fulfilling that predicate. And in addition there are ways of modifying the method to take care of certain cases where even this latter condition is not satisfied. But we have not found any general formulation that will cover all cases regardless of how the individuals concerned overlap one another.

The method will, however, help us in finding a nominalistic reduction for even so platonistic[10]-sounding a statement as:

There are more age-classes than grade-classes in the White School.

We just replace this by:

There are more age-wholes than grade-wholes in the White School,

where an age-whole is the *individual* composed of all pupils in the school who were born during a single calendar year, and a grade-whole is an individual com-

[10] We use 'platonistic' as the antithesis of 'nominalistic.' Thus any language or theory that involves commitment to any abstract entity is platonistic.

posed of all pupils who receive equally advanced instruction. The new sentence is then readily translated in the same way as the one about cats and dogs.

A combination of devices already described enables us to translate a statement like:

There are exactly one-third as many Canadians as Mexicans.

Letting 'the Mexican whole' stand for the individual that is comprised of all Mexicans, the translation runs:

There are some mutually discrete wholes x, y, and z such that each is comprised of Mexicans and such that $x + y + z =$ the Mexican whole; and there are exactly as many Canadians (in all) as there are Mexicans in x and as in y and as in z. The last clause may then be further translated by a slight variation of the method used in the example of cats and dogs.

The foregoing samples will illustrate some of the means that remain in our hands for interpreting statements that *prima facie* have to do with abstract entities. Certainly we have not as yet reached our goal of knowing how to deal with every statement we are not ready to dispense with altogether. But there is as yet no convincing reason for supposing the goal unattainable. Some of the devices used above are rather powerful, and by no means all the possible methods have been explored.

Since, however, we have not as yet discovered how to translate all statements that we are unwilling to discard as meaningless, we describe in following sections a course that enables us—strictly within the limitations of our language and without any retreat from our position—to talk *about* certain statements without being able to translate them.

5. Elements of nominalistic syntax. It may naturally be asked how, if we regard the sentences of mathematics merely as strings of marks without meaning, we can account for the fact that mathematicians can proceed with such remarkable agreement as to methods and results. Our answer is that such intelligibility as mathematics possesses derives from the syntactical or metamathematical rules governing those marks. Accordingly we shall try to develop a syntax language that will treat mathematical expressions as concrete objects—as actual strings of physical marks.[11] Since one mark is as concrete as another, we can deal with such marks and strings as 'ϵ' and '$(v)(v \epsilon v \mid v \epsilon v)$' quite as well as with ones like '(' or 'Eiffel Tower.' But our syntax language must itself be purely nominalistic; it must make no use of terms or devices which involve commitment to abstract entities. It might seem that this program could be carried out without any difficulty once we have specified that we are dealing with concrete marks; but actually classical syntax has depended so heavily upon platonistic devices in constructing its definitions that the nominalist is faced with the necessity of finding new means of definition at almost every step. Not only subsidiary terms,

[11] We might, equally consistently with nominalism, construe marks phenomenally, as events in the visual (or in the auditory or tactual) field. Moreover, although we shall regard an appropriate object during its entire existence as a single mark, we could equally well—and even advantageously if we want to increase the supply of marks—construe a mark as comprising the object in question during only a single moment of time.

but such key terms as 'formula,' 'substitution,' and 'theorem' have to be defined by quite new routes.[12]

The platonistic object language that our nominalistic syntax is to treat of must contain notations for truth-functions, quantification, and membership. All we need for these purposes are parentheses, variables, the stroke '|' of alternative denial, and the sign 'ϵ' of membership. Parentheses will serve both for enclosing alternative denials to indicate groupings and for enclosing variables to form universal quantifiers. To simplify our syntactical treatment, let us require that each alternative denial be enclosed in parentheses—even when it stands apart from any broader context. As variables we may use 'v', 'v'', 'v''', etc., so that the simple typographical shapes of the object language reduce to six: 'v', '$'$', '(', ')', '|', and 'ϵ'.

As already mentioned, the characters of our language are not these abstract shapes—which we, as nominalists, cannot countenance—but rather concrete marks or inscriptions. We can, however, apply shape-predicates to such individuals; thus 'Vee x' will mean that the object x is a vee (i.e., a 'v'-shaped inscription), and 'Ac x' will mean that x is an accent (i.e., a '$'$'-shaped inscription), and 'LPar x' will mean that x is a left parenthesis, and 'RPar x' will mean that x is a right parenthesis, and 'Str x' will mean that x is a stroke (a '|'-shaped inscription), and 'Ep x' will mean that x is an epsilon.

But it happens actually that left parentheses and right parentheses are alike in shape, and distinguishable only by their orientation in broader contexts. It would appear therefore that instead of writing 'LPar x', to mean that x is intrinsically a left parenthesis, we should write 'LPar xy', meaning that x is a left parenthesis from the point of view of its orientation within the longer inscription y; and correspondingly for 'RPar.' Since however this exceptional treatment is made necessary solely by a typographical idiosyncrasy, we may disregard it. The reader may, if he likes, restore an intrinsic distinction between left and right parentheses by thinking of each left parenthesis as comprising within itself the straight uninked line joining its tips.

Our nominalistic syntax must contain, besides the six shape-predicates, some means of expressing the *concatenation* of expressions. We shall write 'Cxyz' to mean that y and z are composed of whole characters of the language, in normal orientation to one another, and contain neither split-off fragments of characters nor anything extraneous, and that the inscription x consists of y followed by z. The characters comprising y and z may be irregularly spaced; furthermore the inscription x will be considered to consist of y followed by z no matter what the spatial interval between y and z, provided that x contains no character that occurs in that interval.

The two remaining primitives of our syntax language are abbreviations of the familiar predicates 'is part of' and 'is bigger than.' 'Part xy' means that x,

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[12] The idea of dealing with the language of classical mathematics in terms of a nuclear syntax language that would meet nominalistic demands was suggested in 1940 by Tarski. In the course of that year the project was discussed among Tarski, Carnap, and the present writers, but solutions were not found at that time for the technical problems involved.

whether or not it is identical with y, is contained entirely within y. 'Bgr xy' means that x is spatially bigger than y.

Our syntax language, then, contains the nine predicates 'Vee', 'Ac', 'LPar', 'RPar', 'Str', 'Ep', 'C', 'Part', and 'Bgr', together with variables, quantifiers and the usual truth-functional notations '$\vee$', '.', etc. The variables take as values any concrete objects.

6. Some auxiliary definitions. We now proceed to define certain useful auxiliary predicates. First, it is convenient to have four-, five-, and six-place predicates of concatenation. The definitions are obvious:

D1. $$\mathrm{C}xyzw = (\exists t)(\mathrm{C}xyt \,.\, \mathrm{C}tzw),$$ [13]

D2. $$\mathrm{C}xyzwu = (\exists t)(\mathrm{C}xyt \,.\, \mathrm{C}tzwu),$$

D3. $$\mathrm{C}xyzwus = (\exists t)(\mathrm{C}xyt \,.\, \mathrm{C}tzwus).$$

Also, later definitions will be shortened considerably if we can say briefly that a given individual is a *character* of our object language. Since a character is any concrete object that is either a vee or an accent or a left parenthesis or etc., the definition runs:

D4. $$\mathrm{Char}\, x =.\ \mathrm{Vee}\, x \vee \mathrm{Ac}\, x \vee \mathrm{LPar}\, x \vee \mathrm{RPar}\, x \vee \mathrm{Str}\, x \vee \mathrm{Ep}\, x.$$

Convenience is similarly served by the definition of an *inscription* as an object composed of whole characters in normal orientation to one another. In view of the interpretation of 'C' in §5, the definition is easy:

D5. $$\mathrm{Insc}\, x =.\ \mathrm{Char}\, x \vee (\exists y)(\exists z)\mathrm{C}xyz.$$

An inscription x is said to be an *initial segment* of another, y, if x is identical with y or there is some inscription z such that y consists of x followed by z.

D6. $$\mathrm{InitSeg}\, xy =.\ \mathrm{Insc}\, x \,.\, x = y \,.\!\vee (\exists z)\mathrm{C}yxz.$$

The definition of *final segment* is strictly parallel:

D7. $$\mathrm{FinSeg}\, xy =.\ \mathrm{Insc}\, x \,.\, x = y \,.\!\vee (\exists z)\mathrm{C}yzx.$$

An inscription x is said to be a *segment* of y if x is an initial segment of some final segment of y.

D8. $$\mathrm{Seg}\, xy = (\exists z)(\mathrm{InitSeg}\, xz \,.\, \mathrm{FinSeg}\, zy).$$

A segment x—whether initial, final, or interior—of an inscription y will be continuous relative to y, in the sense that if x contains two characters of y then

[13] The sign '=', when it occurs as the main connective in definitions in this paper, is not to be thought of as expressing identity. It is to be regarded rather as constituting, in combination with the 'D' which precedes each definition-number, a mark of definitional abbreviation; and it may occur between name-matrices and statement-matrices indifferently. The definition D1 is to be understood as a convention to this effect: '$\mathrm{C}xyzw$' *is to be understood as an abbreviation of* '$(\exists t)(\mathrm{C}xyt \,.\, \mathrm{C}tzw)$'; *and a similar understanding is to obtain when any other variables are used in place of* 'x', 'y', 'z', *and* 'w', *provided that a variable distinct from them is used in place of* 't'. Other definitions are to be construed analogously.

x must contain all the characters which occur in y between those two. The characters of a segment x of y may still be irregularly spaced, but only because of irregular spacing in y itself.

We shall later want to be able to say that two inscriptions are equally long, not in the sense that their ends are equally far apart but in the sense that each inscription has as many characters as the other. Since the characters in any inscription are discrete from one another, this numerical comparison can be handled in a way explained in §4. We begin by so defining 'Bit' for our present purposes that 'Bit x' means that x is just as big as every smallest character.

D9. $\text{Bit } x =. (y)(\text{Char } y \supset \sim\text{Bgr } xy) \,.\, (\exists z)(\text{Char } z \,.\, \sim\text{Bgr } zx).$

It must not be supposed that, because accents are in general the smallest characters of our object language, every accent will be a bit. For accents may vary in size, and only the smallest characters, along with everything that is just as big, will be bits.

An inscription x is *longer than* another, y, if x contains more characters than y. Using the same method as for the example of cats and dogs in §4—where a verbal explanation is given—we define:

D10. $\text{Lngr } xy =. \text{Insc } x \,.\, \text{Insc } y \,.\, (z)\{(w)[\text{Char } w \,.\, \text{Part } wx \,.\!\supset (\exists u)(\text{Bit } u \,.\, \text{Part } uw \,.\, \text{Part } uz)] \supset (\exists t)[(r)(\text{Char } r \,.\, \text{Part } ry \,.\!\supset (\exists s)(\text{Bit } s \,.\, \text{Part } sr \,.\, \text{Part } st)) \,.\, \text{Bgr } zt]\}.$

Two inscriptions are *equally long* if neither is longer than the other.

D11. $\text{EqLng } xy =. \text{Insc } x \,.\, \text{Insc } y \,.\, \sim\text{Lngr } xy \,.\, \sim\text{Lngr } yx.$

We can now define what we shall mean by saying that two inscriptions are *like* one another. Two characters are alike if both are vees, or both are accents, or etc. Two inscriptions x and y are alike if they are equally long and if, for every two equally long inscriptions z and w such that z is an initial segment of x and w is an initial segment of y, the segments z and w end in like characters.

D12. $\text{Like } xy =. \text{EqLng } xy \,.\, (z)(w)\{\text{EqLng } zw \,.\, \text{InitSeg } zx \,.\, \text{InitSeg } wy \,.\!\supset (\exists s)(\exists t)(\text{FinSeg } sz \,.\, \text{FinSeg } tw : \text{Vee } s \,.\, \text{Vee } t \,.\!\vee\!.\, \text{Ac } s \,.\, \text{Ac } t \,.\!\vee\!.\, \text{LPar } s \,.\, \text{LPar } t \,.\!\vee\!.\, \text{RPar } s \,.\, \text{RPar } t \,.\!\vee\!.\, \text{Str } s \,.\, \text{Str } t \,.\!\vee\!.\, \text{Ep } s \,.\, \text{Ep} t)\}.$

Note that only inscriptions can be "alike", in the sense here defined, since only inscriptions can be equally long; and further, that likeness depends solely upon the component characters and their order of occurrence, not upon identical spacing.

7. Variables and quantification. A variable of our object language is a vee, or a vee together with a string of one or more accents following it. We first define a *string of accents* as any inscription of which every part that is a character is an accent.

D13. $\text{AcString } x =. \text{Insc } x \,.\, (z)(\text{Part } zx \,.\, \text{Char } z \,.\!\supset \text{Ac } z).$

The definition of a *variable* is then readily formulated.

D14. Vbl x =. Vee $x \vee (\exists y)(\exists z)(\text{Vee } y \,.\, \text{AcString } z \,.\, Cxyz)$.

A variable is a vee or the result of concatenating a vee with a string of accents.

A quantifier will be simply a variable in parentheses. But it is more useful to define a *string of* (one or more) *quantifiers* directly. A method for doing this becomes evident when we reflect that any inscription will be a string of quantifiers if it begins and ends with facing parentheses and is such that every pair of facing parentheses within it frames an inscription that is either a variable or contains parentheses back to back.

D15. QfrString $x = (\exists y)(\exists z)\{\text{LPar } y \,.\, \text{RPar } z \,.\, (\exists w)Cxywz \,.\, (s)(t)(u)(k)$ $[\text{LPar } t \,.\, \text{RPar } k \,.\, Cstuk \,.\, \text{Seg } sx \,.\!\supset\!.\, \text{Vbl } u \vee (\exists p)(\exists q)(\exists r)(\text{RPar } q \,.\, \text{LPar } r \,.$ $Cpqr \,.\, \text{Seg } pu)]\}$.

Then let us call x a *quantification of* y if x consists of a string of quantifiers followed by y.

D16. $\qquad$ Qfn $xy = (\exists z)(\text{QfrString } z \,.\, Cxzy)$.

8. Formulas. An atomic formula of the object language consists of two variables with an epsilon between them.

D17. $\qquad$ AtFmla $x = (\exists w)(\exists y)(\exists z)(\text{Vbl } w \,.\, \text{Ep } y \,.\, \text{Vbl } z \,.\, Cxwyz)$.

We are supposing that the class logic to be developed in the object language will use one or another of the alternatives to the theory of types, so that epsilons may grammatically occur between any variables without restriction.

The non-atomic formulas of the object language are constructed from the atomic formulas by quantification and alternative denial. In order to define an *alternative denial* we first need to be able to say that a given inscription x *contains exactly as many left as right parentheses*. This will be the case if x lacks parentheses altogether; and it will be the case also if the inscription which consists of all the left parentheses in x and the inscription which consists of all the right parentheses in x are equally long in the sense of D11. In symbols:

D18. EqPar x =. $(u)(\text{LPar } u \vee \text{RPar } u \,.\!\supset \sim\!\text{Seg } ux) \vee (\exists y)(\exists z)\{\text{EqLng}$ $yz \,.\, (w)(\text{Char } w \supset: \text{LPar } w \,.\, \text{Seg } wx \,.\!\equiv \text{Seg } wy : \text{RPar } w \,.\, \text{Seg } wx \,.\!\equiv \text{Seg } wz)\}$.

Now for an inscription x to be the *alternative denial* of y and z it is necessary that x consist of a left parenthesis followed by y, then a stroke, then z, and finally a right parenthesis. But this is not enough. We must make sure that the beginning and ending parentheses are "mates"—that is, that they are paired with each other and not with other parentheses that occur between them. Also we must make sure that the stroke between y and z is the main connective in x. We can accomplish all this by requiring that y contain an equal number of left and right parentheses, and similarly for z, but that this be true of no initial segment of x (except x itself).

D19. ADxyz =. EqPar y . EqPar z . $(r)(s)(Cxrs \supset. \sim\!\text{EqPar } r) \,.\, (\exists t)(\exists u)(\exists w)$ $(\text{LPar } t \,.\, \text{Str } u \,.\, \text{RPar } w \,.\, Cxtyuzw)$.

The *formulas* of the object language comprise the atomic formulas and every inscription constructed from them by means of quantification and alternative denial. Some ways in which one might naturally seek to reduce this to a formal definition are not feasible in a nominalistic syntax.[14] Our method is to begin by defining a *quasi-formula* as anything which is an atomic formula, an alternative denial, or a quantification of an atomic formula or alternative denial.

D20. QuasiFmla x = $(\exists y)(x = y \,.\!\vee$ Qfn xy : AtFmla $y \vee (\exists w)(\exists z)ADywz)$.

A quasi-formula will not necessarily be a formula, since the components of the alternative denial are not required to be formulas. But in terms of this notion of quasi-formula we can now easily define *formula:*

D21. Fmla x =. QuasiFmla x . $(w)(y)(z)($ADwyz . Seg wx .$\supset$. QuasiFmla y . QuasiFmla $z)$.

In other words, a formula is a quasi-formula such that every alternative denial in it is an alternative denial of quasi-formulas.

By requiring even the shortest alternative denials in a formula x to be alternative denials of quasi-formulas, the definition requires them to be alternative denials of atomic formulas or of quantifications of atomic formulas, and this makes them genuine formulas in the intuitively intended sense of the word. Accordingly, by requiring also the next more complex alternative denials in x to be alternative denials of quasi-formulas, the definition guarantees that these also will be formulas in the intuitively intended sense; and so on, to x itself.

9. Axioms and rules. Now that we have specified the characters and formulas of the object language within our nominalistic syntax language, the next problem is to describe the sorts of notational operations which pass for logical *proof* among the users of that object language. A full solution of this problem would consist in the formulation, in our syntax language, of a condition which is necessary and sufficient in order that an inscription x be a *theorem* of the object logic.

The theorems are those formulas of the object language which follow from certain axioms by certain rules of inference. The axioms should be so chosen that we can obtain from them, by the rules of inference, every formula which is valid according to the logic of alternative denial and quantification and, in addi-

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[14] Using essentially the method of Frege's definition of the ancestral of a relation, we might say that x is a formula if it belongs to every class which contains all atomic formulas and all quantifications and alternative denials of its members. But this definition is unallowable because of its use of quantification over classes; cf. §4.—There is indeed a completely general method, in syntax, of deriving ancestrals and kindred constructions without appeal to classes of expressions. This is the method of "framed ingredients" which appears in Quine, ***Mathematical logic***, §56. The method consists essentially of these two steps: (1) the Frege form of definition is so revised that the classes to which it appeals can be limited to finite classes without impairing the result; (2) finite classes of expressions are then identified with *individual* expressions wherein the "member"-expressions occur merely as *parts* marked off in certain recognizable ways. However, when as nominalists we conceive of expressions strictly as concrete inscriptions, we find the method of framed ingredients unsatisfactory, because its success depends too much on what inscriptions happen to exist in the world. Actually, though, the nominalistic definition of proof in the present paper will be simpler than that in terms of framed ingredients; for it will not require the lines of a proof to be concatenated, nor to be marked off by intervening signs.

tion, a goodly array of formulas whose alleged validity is supposed to proceed from special properties of class-membership. We cannot aspire to completeness in this last regard, in view of Gödel's result.

There are many essentially equivalent sets of axioms suitable to the above purposes. The axioms which we shall adopt fall under three heads: *axioms of alternative denial, axioms of quantification*, and *axioms of membership*. In setting them forth let us understand '$\sim \cdots$' as short for '$(\cdots \mid \cdots)$'.

Axioms of alternative denial: All formulas of the form:

$$((P \mid (Q \mid R)) \mid ((S \mid \sim S) \mid ((S \mid Q) \mid \sim(P \mid S)))),$$ [15]

like letters being replaced by like formulas.

Axioms of quantification: All formulas of the forms:

(1) $((v)(P \mid \sim Q) \mid \sim((v)P \mid \sim(v)Q))$,

(2) $(R \mid \sim(v)R)$ (where 'v' is not free in 'R'),

(3) $((v)P \mid \sim S)$ (where 'S' is the result of substituting some variable for 'v' in 'P').

If the reader reflects that the sign-combination '$\mid \sim$' amounts to '$\supset$', he will recognize in the forms (1) – (3) a familiar set of axiom-schemata for quantification theory.[16] Like capitals in (1) – (3) are of course to be understood as replaced by like formulas, and the vees by like variables. The two brief provisos appended to (2) and (3), above, may be stated more precisely as follows: (i) the formulas supplanting the 'R's contain no free variables like the variables supplanting the vees, and (ii) the formula supplanting the 'S' is like the formula supplanting the 'P' except perhaps for containing other free variables, like one another, in place of all free variables like the variable supplanting the vee.

Axioms of membership: Here it happens that a limited list of specific expressions is adequate; e.g., Hailperin's.[17] Let us suppose such a list put over into the primitive notation of our object language and set down here; then our axioms of membership are all inscriptions like those in the list.

In addition to the axioms, we need two *rules of inference:*

(1) From any formula, together with the result of putting a formula like it for 'P' and any formulas for 'Q' and 'R' in '$(P \mid (Q \mid R))$', infer any formula like the one which was put for 'Q'.[18]

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[15] This is Łukasiewicz's simplification of Nicod's axiom schema. See Jan Łukasiewicz, *Uwagi o aksyomacie Nicod'a i o "dedukcyi uogólniającej"*, ***Księga pamiątkowa Polskiego Towarzystwa Filozoficznego we Lwowie***, 1931, pp.2–7; also Jean Nicod, *A reduction in the number of primitive propositions of logic*, ***Proceedings of the Cambridge Philosophical Society***, vol. 19 (1917–20), pp. 32–41.

[16] They answer to 4.4.4, 4.4.5, and 4.4.6 of F. B. Fitch, *The consistency of the ramified Principia*, this JOURNAL, vol. 3 (1938), pp. 140–149; also to *102–*104 of W. V. Quine, ***Mathematical logic***, p. 88.

[17] Theodore Hailperin, *A set of axioms for logic*, this JOURNAL, vol. 9 (1944), pp. 1–19.

[18] This is Nicod's generalization of *modus ponens*; see footnote 15.

(2) From any formula infer any quantification thereof.

To reach a definition of 'Axiom' we must first be able to define what it means to be an axiom of any given one of the five kinds above described. A simple auxiliary definition will be useful:

D22. $\mathrm{D}xy = (\exists z)(\mathrm{Like}\ yz\ .\ \mathrm{AD}xyz)$;

i.e., that x is a *denial* of y means that x is the alternative denial of y and some other inscription exactly like y.

Definition of 'AADx', meaning that x is an *axiom of alternative denial,* is achieved by stating formally what we can observe from the general schema already given: that every axiom of alternative denial is an alternative denial of two formulas; one of these two main components is an alternative denial of formulas of which one is an alternative denial of formulas; the other of the two main components is an alternative denial of formulas of which one is an alternative denial of a formula with a formula like the denial of that formula, while the other is . . . etc., etc. In symbols:

D23. $\mathrm{AAD}x = (\exists f)(\exists g)(\exists h)(\exists i)(\exists j)(\exists k)(\exists l)(\exists m)(\exists n)(\exists p)(\exists q)(\exists r)(\exists s)(\exists t)(\exists u)(\exists w)(\exists y)(\exists z)(\mathrm{Fmla}\ f\ .\ \mathrm{Fmla}\ g\ .\ \mathrm{Fmla}\ h\ .\ \mathrm{Fmla}\ i\ .\ \mathrm{Like}\ ki\ .\ \mathrm{Like}\ lg\ .\ \mathrm{Like}\ mf\ .\ \mathrm{Like}\ ni\ .\ \mathrm{AD}pgh\ .\ \mathrm{AD}qfp\ .\ \mathrm{D}ri\ .\ \mathrm{AD}sir\ .\ \mathrm{AD}tkl\ .\ \mathrm{AD}umn\ .\ \mathrm{D}wu\ .\ \mathrm{AD}ytw\ .\ \mathrm{AD}zsy\ .\ \mathrm{AD}xqz)$.

Formulation of 'AQ1 x', meaning that x is an axiom of quantification of kind (1), proceeds in the same way; we shall omit the definition.

Formulation of 'AQ2 x' offers the one additional difficulty that in order to express stipulation (i), appearing in the above description of the axioms of quantification, we must have a definition of *free variable.* A variable x is a free variable in an inscription y if x is a segment of y not followed by any additional accents in y, and if furthermore x is not a segment of any segment of y that consists of a formula preceded by a quantifier consisting of a variable like x framed in parentheses.

D24. $\mathrm{Free}\ xy =.\ \mathrm{Vbl}\ x\ .\ \mathrm{Seg}\ xy\ .\ (z)(w)(\mathrm{Ac}\ w\ .\ \mathrm{C}zxw\ .\ \sim\mathrm{Seg}\ zy)\ .\ (q)(r)(s)(t)(u)(\mathrm{LPar}\ q\ .\ \mathrm{Like}\ rx\ .\ \mathrm{RPar}\ s\ .\ \mathrm{Fmla}\ t\ .\ \mathrm{C}uqrst\ .\ \mathrm{Seg}\ uy\ .\supset \sim\mathrm{Seg}\ xu)$.

The definition of 'AQ2 x' is then quite straightforward and may be omitted here.

Formulation of 'AQ3 x' offers a further complication for nominalistic syntax. The problem lies in the notion of *substitution,* involved in stipulation (ii). Let z and w be the respective formulas supplanting the 'P' and 'S' of (3), let y be the variable supplanting the 'v', and let x be like the free variables which are to appear in w in place of the free variables like y in z. We have to find a way within nominalistic syntax of defining 'Subst $wxyz$,' meaning that *the formula w is like the formula z except for having free variables like x wherever z contains free variables like y.* Our method of definition depends upon the fact that the condition in the foregoing italics is equivalent to the following one: *What remains when all free variables like y are omitted from the formula z is like what remains when some free variables like x are omitted from the formula w.* The formal definition is as follows:

D25. Subst $wxyz$ =. Fmla w . Fmla z . $(\exists t)(\exists u)\{$Like tu . $(s)[$Char s $\supset$: $(r)($Like ry . Free rz .$\supset$ $\sim$Seg $sr)$. Seg sz .$\equiv$ Seg su : $(r)($Like rx . Free rw .$\supset$ $\sim$Seg $sr)$.$\supset$. Seg sw $\equiv$ Seg $st]\}$.

It was largely for the purpose of this definition that we so defined likeness of inscriptions as to allow their characters to be differently spaced.

Now that this definition is accomplished, the definition of 'AQ3 x' offers no further difficulty (and is omitted here).

Definition of axioms of the fifth and final kind—the axioms of membership, "AM"—presents no problem; we can specify them in our syntax simply by spelling them out explicitly with the help of our primitive predicates.

We are then ready for a general definition of what it means for x to be an *axiom* of our object language. It means simply that x is an axiom of one of the five kinds specified.

D26. Axiom x =. AADx $\vee$ AQ1 x $\vee$ AQ2 x $\vee$ AQ3 x $\vee$ AMx.

An inscription x is called an *immediate consequence* of inscriptions y and z just in case x follows from y and z by one application of rule of inference (1), or from y by rule of inference (2).

D27. ICxyz =. $(\exists u)(\exists w)($ADuxw . ADyzu $\vee$ AD$zyu)$ $\vee$ Qfn xy.

10. Proofs and theorems. An inscription is a theorem if it has a proof; and a proof is constructed by a series of steps of immediate consequence, starting from axioms. Roughly, a proof is describable as composed of one or more lines such that each is either an axiom or an immediate consequence of preceding lines. Actually we need not require that the so-called "lines" of a proof be at different levels on a page, or be segregated from one another by any other device. They could even be written end to end without intervening punctuation, and we could still single them out uniquely as separate "lines." For, the grammar of the object language is such that the result of directly concatenating two formulas z and w will never be a segment of a larger formula, nor will it contain as segments any formulas other than those which are segments of z alone or w alone. Accordingly it will be convenient in general to speak of x as a *line* of y (where y may or may not be a proof) if x is a formula which is part of y but not part of any other formula in y.

D28. Line xy = $(z)($Fmla z . Part xz . Part zy .$\equiv$. $z = x)$.

If a theorem is to be defined as a formula for which a proof exists, it is important not to demand that all lines of the proof be assembled in proper order in any one place and time. Accordingly we shall so define a proof as to allow it to consist of lines wherever they may be—perhaps scattered at random throughout the universe, and perhaps not even all existing at any one moment or within any one century.

According to the rough characterization of proof proposed two paragraphs back, each line must be either an axiom or an immediate consequence of *preceding* lines. The reason for the word 'preceding' here is to rule out cases where every line is deducible from other lines, in circular fashion, while not all lines are deducible ultimately from axioms. However, we must now resort to some other expedient for excluding such circularity; for we have chosen to dispense with the ordering of lines of a proof, and this deprives us of the notion of a "preceding" line.

An expedient which will be shown to meet the requirements is this: We stipulate that if any individual y contains as parts some lines of a proof x but none which are axioms, then some line of x which lies in y must be an immediate consequence of lines of x which lie outside y. The following, then, is our definition:

D29. Proof $x = (y)\{(\exists z)(\text{Line } zx \,.\, \text{Part } zy) \,.\, (w)(\text{Axiom } w \,.\, \text{Line } wx \,.\!\supset \sim\!\text{Part } wy) \,.\!\supset (\exists s)(\exists t)(\exists u)(\text{Line } sx \,.\, \text{Part } sy \,.\, \text{Line } tx \,.\, \sim\!\text{Part } ty \,.\, \text{Line } ux \,.\, \sim\!\text{Part } uy \,.\, \text{IC}stu)\}$.

In order to establish that this definition is adequate to our purposes, we shall now show (1) that if x is a "proof" in the sense of D29, then we can specify an order of "precedence" among the lines of x such that every line is either an axiom or an immediate consequence of "earlier" lines; and we shall also show conversely that (2) if x is such that an order of precedence of the above kind can be specified among its lines, then x is a "proof" in the sense of D29.

(1) is established as follows. Suppose x is a "proof" in the sense of D29. We can begin our specification of an order of precedence among the lines of x by picking out, in an arbitrary order $L_1, L_2, \cdots, L_k$, all those lines of x which are axioms. Next, from among the remaining lines of x, we pick one, call it L_{k+1}, which is an immediate consequence of lines from among $L_1, L_2, \cdots, L_k$. (There will be such a line; for, by D29, that individual y which consists of all lines of x except $L_1, L_2, \cdots, L_k$ must contain a line which is an immediate consequence of lines of x outside y.) Next, from among the remaining lines of x, we pick one—call it L_{k+2}—which is an immediate consequence of lines from among $L_1, L_2, \cdots, L_{k+1}$. (There will be such a one, for the same reason as before.) Continuing thus, we eventually specify an order of precedence of the required kind.

(2) is established as follows. Suppose the lines of x can be counted off in some order such that each line is an axiom or an immediate consequence of earlier lines. Now consider anything y which contains some lines of x but none which are axioms. From among those lines of x which are parts of y, pick out the one which is earliest according to the assumed order. It must be either an axiom or an immediate consequence of earlier lines of x. But it is not an axiom, for y contains none of the lines of x which are axioms. Hence it is an immediate consequence of earlier lines of x; and those earlier lines are not in y. We see therefore that y contains a line of x which is an immediate consequence of lines of x outside y. Since y was taken as *any* individual containing some lines of x but none which are axioms, it follows that x is a proof in the sense of D29.

So it is now clear that D29, without stipulating any order among lines, gives us an adequate version of 'proof.'

Note incidentally that D29 abstains even from any requirement that a proof consist wholly of formulas; the "lines" of a proof x are indeed formulas, but x

may contain also any manner of additional debris without ill effect. Proofs are not in general "inscriptions," in the sense of D5.

If a theorem is any inscription for which there is a proof, then an inscription is a theorem if and only if it is a line of some proof. But this formulation is a little too narrow. Given any inscription y for which a proof x exists, it will be true that for each inscription z that is like y, and that lies outside of x, a proof will also exist, consisting for example of z together with those lines of x that are not identical with y. Hence if y is a theorem all such inscriptions like it will also be theorems. But suppose that some inscription w which is like y lies embedded within some line t in the proof x, and suppose that no other line like t exists; in this case there may be no proof for w, so that some inscriptions like the theorem y may not be theorems. To prevent this anomaly, we construct our definition so that an inscription will be a *theorem* if and only if it is *like* some line of some proof. ('Like' has of course been so defined as to be reflexive.)

D30. $\text{Thm } x = (\exists y)(\exists z)(\text{Proof } y \,.\, \text{Line } zy \,.\, \text{Like } xz).$

With the definition so constructed, it follows that all immediate consequences of theorems are theorems. But some formulas may still fail to qualify as theorems solely because no inscription exists anywhere at any time to stand as a needed intermediate line in an otherwise valid proof. Such limitations would prove awkward if we had to depend upon the accidental existence of inscriptions that are perceptibly marked out against a contrasting background. But we may rather, as suggested earlier (§2), construe inscriptions as all appropriately shaped portions of matter. Then the only syntactical descriptions that will fail to have actual inscriptions answering to them will be those that describe inscriptions too long to fit into the whole spatio-temporally extended universe. This limitation is hardly likely to prove embarrassing. (If we ever should be handicapped by gaps in the proof of an inscription wanted as a theorem, however, we can strengthen our rules of inference to bridge such gaps; for, the number of steps required in a proof depends upon the rules, and the rules we have adopted can be altered or supplemented considerably without violation of nominalistic standards.)

It may be interesting to observe in passing that the theoretical limitations just considered obtain under platonistic syntax as well, if that syntax construes expressions as *shape-classes* of inscriptions; for, shapes having no inscriptions as instances reduce to the null class and are thus identical.[19] The platonist may indeed escape the limitations of concrete reality by hypostatizing an infinite realm of abstract entities—the series of numbers—and then arithmetizing his syntax; the nominalist, on the other hand, holds that any recourse to platonism is both intolerable and unnecessary.

11. Conclusion. In our earlier sections we studied the problem of translating

[19] According to the classical principles of syntax, any two expressions x and y have concatenate $x^\frown y$; and moreover $x^\frown y$ is always distinct from $z^\frown w$, unless the characters occurring in x and in y are successively the same as those in z and in w. This combination of principles is as untenable from the point of view of a platonistic syntax of shape-classes as from the point of view of nominalism.

into nominalistic language certain nonsyntactical sentences which had appeared to be explicable only in platonistic terms. In §§5–10 we have been concerned with giving such a translation for syntax. This syntax enables us to describe and deal with many formulas (of the object language) for which we have no direct nominalistic translation. For example, the formula which is the full expansion in our object language of '$(n)(n + n = 2n)$' will contain variables calling for abstract entities as values; and if it cannot be translated into nominalistic language, it will in one sense be meaningless for us. But, taking that formula as a string of marks, we can determine whether it is indeed a proper formula of our object language, and what consequence-relationships it has to other formulas. We can thus handle much of classical logic and mathematics without in any further sense understanding, or granting the truth of, the formulas we are dealing with.

The gains which seem to have accrued to natural science from the use of mathematical formulas do not imply that those formulas are true statements. No one, not even the hardiest pragmatist, is likely to regard the beads of an abacus as true; and our position is that the formulas of platonistic mathematics are, like the beads of an abacus, convenient computational aids which need involve no question of truth. What is meaningful and true in the case of platonistic mathematics as in the case of the abacus is not the apparatus itself, but only the description of it: the rules by which it is constructed and run. These rules we do understand, in the strict sense that we can express them in purely nominalistic language. The idea that classical mathematics can be regarded as mere apparatus is not a novel one among nominalistically minded thinkers; but it can be maintained only if one can produce, as we have attempted to above, a syntax which is itself free from platonistic commitments.

At the same time, every advance we can make in finding direct translations for familiar strings of marks will increase the range of the meaningful language at our command.

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